

THE K -THEORY OF FINITELY MANY COMMUTING ENDOMORPHISMS

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ABSTRACT. For a field k we compute the K -theory of the exact category of $k[t_1, \dots, t_n]$ -modules that are finite-dimensional over k , generalising the work of Kelley and Spanier.

1. INTRODUCTION

Let A be a commutative ring and $\mathcal{P}(A)$ the category of finitely-generated projective A -modules. Let $\text{End } \mathcal{P}(A)$ be the category whose objects are endomorphisms $P \rightarrow P$ where $P \in \mathcal{P}(A)$, and whose morphisms $\alpha : (P \rightarrow P) \rightarrow (Q \rightarrow Q)$ are morphisms $\alpha : P \rightarrow Q$ in $\mathcal{P}(A)$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & Q \\ \downarrow & & \downarrow \\ P & \xrightarrow{\alpha} & Q \end{array}$$

commutes. The category $\text{End } \mathcal{P}(A)$ is naturally equivalent to the category of $A[t]$ -modules that are finitely generated and projective as A -modules via the inclusion map $A \rightarrow A[t]$. The category $\text{End } \mathcal{P}(A)$ is an exact category and one would like to calculate its Quillen K -theory groups. This type of K -theory calculation falls under the more general problem of calculating for a ring homomorphism $R \rightarrow S$, the K -theory of the category of S -modules that are finitely generated and projective as R -modules.

The calculation of $K_i(\text{End } \mathcal{P}(A))$ was given by Kelley and Spanier [KS68] when A is a field and $i = 0$. Almkvist [Alm78] did the calculation when A is an arbitrary commutative ring and $i = 0$, and when A is a field and i is arbitrary. To describe these computations, define the abelian group whose underlying set is

$$\tilde{A}_0 = \left\{ \frac{1 + a_1 t + \dots + a_n t^n}{1 + b_1 t + \dots + b_m t^m} : a_i, b_j \in A \right\}$$

and whose binary operation is given by the usual multiplication of rational functions. If $f : M \rightarrow M$ is an endomorphism of a finitely-generated projective A -module, then the characteristic polynomial $\lambda_t(f)$ may be defined by extending f to an endomorphism of a free module, and then defining $\lambda_t(f) = \det(1 + tf)$; see [Alm78] for an alternative definition. We can use this map to describe the results of Kelley and Spanier and Almkvist:

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1.1. Theorem. *There is an isomorphism*

$$K_0(\text{End } \mathcal{P}(A)) \longrightarrow K_0(A) \times \tilde{A}_0$$

given on generators by

$$[M \xrightarrow{f} M] \longmapsto ([M], \lambda_t(f)).$$

In this paper, we generalise this result to the case of finitely many commuting endomorphisms and where $A = k$ is a field:

1.2. Theorem. *The algebraic K -groups of the exact category $\text{End}_n \mathcal{P}(k)$ are given by*

$$(1) \quad K_i(\text{End}_n \mathcal{P}(k)) \cong \bigoplus_M K_i(k[t_1, \dots, t_n]/M)$$

where M ranges over all the maximal ideals of the polynomial ring $k[t_1, \dots, t_n]$.

We remark that our proof is different than the proofs in [Alm78] and in [KS68]. One feature is that our result for $i = 0$ is not phrased in terms of a characteristic polynomial map, which is an advantage in the sense that it is more explicit in terms of the structure of the K -groups, but a disadvantage in the sense that working with products is more difficult. The result for $n = 1$ and higher K -theory is [Alm78, Theorem 5.2], but again, our proof is entirely different.

2. THE CATEGORY $\text{End}_n \mathcal{P}(k)$

Let $\text{End}_n \mathcal{P}(k)$ denote the exact category of $k[T] := k[t_1, \dots, t_n]$ -modules that are finitely generated as k -modules. In this section we calculate $K_i(\text{End}_n \mathcal{P}(k))$. A finite-dimensional k -vector space V with any n commuting endomorphisms f_1, f_2, \dots, f_n of V may also be considered as a $k[t_1, \dots, t_n]/I$ module where I is the kernel of the map $k[t_1, \dots, t_n] \rightarrow k[f_1, \dots, f_n]$ given by $t_i \mapsto f_i$; hence:

2.1. Proposition. *The category $\text{End}_n \mathcal{P}(k)$ is naturally equivalent to the filtered direct limit*

$$(2) \quad \varinjlim_I (k[T]/I\text{-}\mathbf{mod}_{\mathbf{fg}})$$

of exact categories, where I runs over the set of ideals of $k[T]$ such that the quotient $k[T]/I$ is finite-dimensional over k , and if $I \subseteq J$ are two such ideals, then the functor $k[T]/J\text{-}\mathbf{mod}_{\mathbf{fg}} \rightarrow k[T]/I\text{-}\mathbf{mod}_{\mathbf{fg}}$ is the forgetful functor induced by the quotient map $k[T]/I \rightarrow k[T]/J$.

Proof. The limit is indeed filtered, since $k[T]/(I \cap J)$ is finite-dimensional whenever $k[T]/I$ and $k[T]/J$ are finite-dimensional. Define a functor

$$F : \varinjlim_I (k[T]/I\text{-}\mathbf{mod}_{\mathbf{fg}}) \rightarrow \text{End}_n \mathcal{P}(A)$$

as follows. Any element in $\varinjlim_I (k[T]/I\text{-}\mathbf{mod}_{\mathbf{fg}})$ is represented by $V \in k[T]/I\text{-}\mathbf{mod}_{\mathbf{fg}}$ for some I . We let $F(V)$ to be the $k[T]$ module given by the forgetful functor induced

by the map $k[T] \rightarrow k[T]/I$. This functor is well-defined because $k[T]/I$ is finite-dimensional, and it is easy to see that F gives the required natural equivalence. \square

Using this observation and the result that taking K -groups of exact categories commutes with filtered direct limits [Qui72, §2], we obtain the following corollary.

2.2. Corollary. *The K -theory of the category $\text{End}_n \mathcal{P}(A)$ may be calculated as the direct limit*

$$(3) \quad K_i(\text{End}_n \mathcal{P}(A)) \cong \varinjlim_I K_i(k[T]/I\text{-}\mathbf{mod}_{\text{fg}})$$

For any ring R , let $\text{Jac}(R)$ denote the Jacobson radical of R . Any surjective ring homomorphism $R \rightarrow S$ induces a surjective ring homomorphism $R/\text{Jac}(R) \rightarrow S/\text{Jac}(S)$, and a commutative diagram of rings

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/\text{Jac}(R) & \longrightarrow & S/\text{Jac}(S) \end{array}$$

In turn, whenever S is finitely generated as an R -module, we have a commutative diagram of forgetful functors

$$(4) \quad \begin{array}{ccc} R\text{-}\mathbf{mod}_{\text{fg}} & \longleftarrow & S\text{-}\mathbf{mod}_{\text{fg}} \\ \uparrow & & \uparrow \\ R/\text{Jac}(R)\text{-}\mathbf{mod}_{\text{fg}} & \longleftarrow & S/\text{Jac}(S)\text{-}\mathbf{mod}_{\text{fg}} \end{array}$$

In particular, this applies to the ring homomorphisms $k[T]/I \rightarrow k[T]/J$ for $I \subseteq J$ appearing in the direct limit of (2).

2.3. Proposition. *The natural transformation of the direct limits of categories induced by the forgetful functors as in (4) for $R = k[T]/I$ induces an isomorphism algebraic K -groups*

$$\varinjlim_I K_i\left(\frac{k[T]/I}{\text{Jac}(k[T]/I)}\text{-}\mathbf{mod}_{\text{fg}}\right) \xrightarrow{\sim} \varinjlim_I K_i(k[T]/I\text{-}\mathbf{mod}_{\text{fg}}).$$

Proof. For any Artinian ring R , the Jacobson radical $\text{Jac}(R)$ is nilpotent (e.g. [Lam91, Theorem 4.12]), and so devissage [Qui72, §5, Theorem 4] shows that the inclusion $R/\text{Jac}(R)\text{-}\mathbf{mod}_{\text{fg}} \rightarrow R\text{-}\mathbf{mod}_{\text{fg}}$ induces an isomorphism

$$K_i(R/\text{Jac}(R)\text{-}\mathbf{mod}_{\text{fg}}) \rightarrow K_i(R\text{-}\mathbf{mod}_{\text{fg}})$$

(see [Wei13, Page 439] for more details). In particular, this applies to $R = k[T]/I$ where $k[T]/I$ is finite-dimensional over k . \square

2.4. Theorem. *The algebraic K -groups of the exact category $\text{End}_n \mathcal{P}(k)$ are given by*

$$(5) \quad K_i(\text{End}_n \mathcal{P}(k)) \cong \bigoplus_M K_i(k[t_1, \dots, t_n]/M)$$

where M ranges over all the maximal ideals of the polynomial ring $k[t_1, \dots, t_n]$.

Proof. We must calculate the limit

$$(6) \quad \varinjlim_I K_i \left(\frac{k[T]/I}{\text{Jac}(k[T]/I)}\text{-}\mathbf{mod}_{\mathbf{fg}} \right)$$

given in Proposition 2.3. So, fix an ideal I such that $k[T]/I$ is finite-dimensional. Then there are finitely many maximal ideals M_1, \dots, M_k of $k[T]$ that contain I and

$$k[T]/I \cong \bigoplus_{j=1}^k (k[T]/I)_{M_j}$$

via the obvious map (e.g. [GW10, Theorem 5.20]). Here, by $(k[T]/I)_{M_i}$, we abuse notation and mean the localization of $k[T]/I$ away from the maximal ideal $M_i(k[T]/I)$. If J is an ideal containing I , then there is a subset $\{N_1, \dots, N_\ell\}$ of the maximal ideals $\{M_1, \dots, M_k\}$ that contain J and the quotient homomorphism $k[T]/I \rightarrow k[T]/J$ induces the map

$$\bigoplus_{j=1}^k (k[T]/I)_{M_j} \longrightarrow \bigoplus_{j=1}^{\ell} (k[T]/J)_{N_j}$$

which must be the projection map. The induced map on K -groups that fits into the direct limit in (6), by is then the map

$$\bigoplus_{j=1}^{\ell} K_i(k[T]/N_j) \longrightarrow \bigoplus_{j=1}^k K_i(k[T]/M_j).$$

given by the sum of the inclusion maps. Taking the direct limit gives the stated result, once we note that $k[T]/M$ is finite-dimensional over k for any maximal ideal M . \square

In particular, we obtain the following amusing, possibly well-known result.

2.5. Corollary. *For a field k , there is an isomorphism of abelian groups*

$$\tilde{k}_0 := \left\{ \frac{1 + a_1 t + \dots + a_n t^n}{1 + b_1 t + \dots + b_m t^m} : a_i, b_j \in k \right\} \cong \bigoplus_{|\text{Spec}(k[T])| - 1} \mathbb{Z}.$$

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